1.5 Eigenvalues, eigenvectors and diagonalization

Definition 1.5.1 (Eigenvalues and eigenvectors)**.** *Let A be an $n \times n$ matrix. If λ is a complex number*[3] *and $\boldsymbol{\xi}$ is a non-zero*[4] *complex vector such that*

$$A\boldsymbol{\xi} = \lambda\boldsymbol{\xi},$$

$A\vec{0} = \vec{0} = \lambda\vec{0}$

then we say that λ is an **eigenvalue** *of A and $\boldsymbol{\xi}$ is an* **eigenvector** *of A associated with λ.*

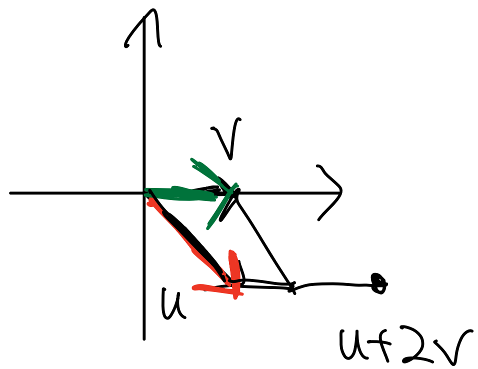
Eigenvector

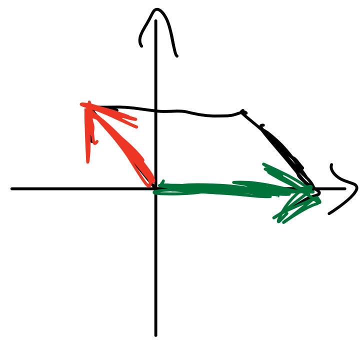
eigenvalue

$$\begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}\underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{u} = \begin{bmatrix}-1\\1\end{bmatrix} = (-1)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}2 & 3\\0 & -1\end{bmatrix}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{v} = \begin{bmatrix}2\\0\end{bmatrix}$$

$$= 2\begin{bmatrix}1\\0\end{bmatrix}$$





$$A(u+2v) = Au + 2Av$$

$$= (-1)u + 2(2v)$$

Definition 1.5.2 (Characteristic polynomial and characteristic equation). *Let A be an $n \times n$ matrix. The* **characteristic polynomial** *of A is the degree n polynomial* $\det(xI - A)$ *in x, where I is the identity matrix. The* **characteristic equation** *of A is the degree n polynomial equation*

$$\det(xI - A) = 0.$$

Find $\vec{\xi}, \lambda$ s.t. $A\xi = \lambda\xi$

$\xi \neq 0$

$A\xi - \lambda\xi = \vec{0}$

$A\xi - \lambda I_n \xi = \vec{0}$

$\underbrace{(A - \lambda I)}\xi = \vec{0}$

want det $= 0 \Rightarrow$ non-zero soln ξ

$B\vec{x} = \vec{0}$

has non-zero soln

iff $\det B = \vec{0}$

Proposition 1.5.3. *Let A be an $n \times n$ matrix.*

1. *A complex number λ is an eigenvalue of A if and only if λ is a root to the characteristic equation* $\det(xI - A) = 0$.

2. *Let λ be an eigenvalue of A. Then $\boldsymbol{\xi}$ is an eigenvector of A associated with λ if and only if $\boldsymbol{\xi} \neq \mathbf{0}$ and $(\lambda I - A)\boldsymbol{\xi} = \mathbf{0}$.*

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Char equation:

$$\det(xI - A) = 0$$

$$\det \begin{bmatrix} x-1 & -1 \\ 2 & x-4 \end{bmatrix} = 0$$

$$(x-1)(x-4) + 2 = 0$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$\lambda = 2, 3$$

For $\lambda = 2$,

$$(2I - A)\xi = \vec{0}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x - y = 0 \\ 2x - 2y = 0 \end{cases}$$

Take $x = y = 1$

$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$$(3I - A)\xi = \vec{0}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x - y = 0 \\ 2x - y = 0 \end{cases}$$

Take $y = 2$, $x = 1$

$$\xi = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1.7 Some transcendental functions

1. **Exponential function**:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \textit{ for } x \in \mathbb{R}$$

2. **Trigonometric functions**: *There are 6 trigonometric functions which are defined as follows.*

Cosine: $$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \textit{ for } x \in \mathbb{R}$$

Sine: $$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \textit{ for } x \in \mathbb{R}$$

Tangent: $$\tan x = \frac{\sin x}{\cos x} \textit{ for } x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$$

Cotangent: $$\cot x = \frac{\cos x}{\sin x} \textit{ for } x \neq k\pi,\ k \in \mathbb{Z}$$

Secant: $$\sec x = \frac{1}{\cos x} \textit{ for } x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$$

Cosecant: $$\csc x = \frac{1}{\sin x} \textit{ for } x \neq k\pi,\ k \in \mathbb{Z}$$

3. **Hyperbolic functions**: *There are 6 hyperbolic functions which are defined as follows.*
Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \text{ for } x \in \mathbb{R}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} \text{ for } x \neq 0$$

Hyperbolic secant:

$$\text{sech} x = \frac{1}{\cosh x} \text{ for } x \in \mathbb{R}$$

Hyperbolic cosecant:

$$\text{csch} x = \frac{1}{\sinh x} \text{ for } x \neq 0$$

Theorem 1.7.2. *The exponential function satisfies*

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

for any $x \in \mathbb{R}$.

Definition 1.7.3 (Logarithmic function)**.** *The* **logarithmic function** *is the function* $\ln : \mathbb{R}^+ \to \mathbb{R}$ *defined for* $x > 0$ *by*

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ *is the inverse function of the exponential function.*

Proposition 1.7.4 (Identities for transcendental functions).

1. *Exponential function:*
   (a) $e^{x+y} = e^x e^y$
   (b) $e^{x-y} = \dfrac{e^x}{e^y}$
   (c) $e^{kx} = (e^x)^k$ *for* $k \in \mathbb{Z}$

2. *Logarithmic function:*
   (a) $\ln(xy) = \ln x + \ln y$
   (b) $\ln \dfrac{x}{y} = \ln x - \ln y$
   (c) $\ln(x^k) = k \ln x$ *for* $k \in \mathbb{Z}$

3. *Trigonometric identities:*
   (a) $\cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$
   (b) $\cos(-x) = \cos x; \quad \sin(-x) = -\sin x; \quad \tan(-x) = -\tan x$
   (c) $\cos(x+y) = \cos x \cos y - \sin x \sin y;$
   $\sin(x+y) = \sin x \cos y + \cos x \sin y;$
   $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
   (d) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$\csc^2 x - \cot^2 x = 1$

4. *Hyperbolic identities:*
   (a) $\cosh^2 x - \sinh^2 x = 1; \quad \operatorname{sech}^2 x + \tanh^2 x = 1; \quad \coth^2 x - \operatorname{csch}^2 x = 1$
   (b) $\cosh(-x) = \cosh x; \quad \sinh(-x) = -\sinh x; \quad \tanh(-x) = -\tanh x$
   (c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y;$
   $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y;$
   $$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$
   (d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x;$
   $\sinh 2x = 2 \sinh x \cosh x;$
   $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

Proposition 1.7.5 (Derivatives of transcendental functions).

1. *Exponential function:*
$$\frac{d}{dx}e^x = e^x$$

2. *Logarithmic function:*
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

3. *Trigonometric functions:*
$$\frac{d}{dx}\cos x = -\sin x; \quad \frac{d}{dx}\sin x = \cos x;$$
$$\frac{d}{dx}\tan x = \sec^2 x; \quad \frac{d}{dx}\cot x = -\csc^2 x;$$
$$\frac{d}{dx}\sec x = \sec x\tan x; \quad \frac{d}{dx}\csc x = -\csc x\cot x$$

4. *Inverse trigonometric functions*[7]*:*
$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}};$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\frac{\pi}{4} = \cos^{-1}\frac{1}{\sqrt{2}}$$

5. *Hyperbolic functions:*
$$\frac{d}{dx}\cosh x = \sinh x; \quad \frac{d}{dx}\sinh x = \cosh x;$$
$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x; \quad \frac{d}{dx}\coth x = -\mathrm{csch}^2 x;$$
$$\frac{d}{dx}\mathrm{sech} x = -\mathrm{sech} x\tanh x; \quad \frac{d}{dx}\mathrm{csch} x = -\mathrm{csch} x\coth x$$

6. *Inverse hyperbolic functions*[8]*:*
$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}};$$
$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}};$$
$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

Proposition 1.7.6 (Integrals of transcendental functions).

$u = \sin x$

1. *Exponential function:*

$$\int e^x dx = e^x + C$$

$$\int \cot x dx$$

2. *Logarithmic function:*

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\ln \frac{1}{x} = -\ln x$$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \int \frac{d \sin x}{\sin x} = \ln|\sin x| + C$$

3. *Trigonometric functions:*

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C;$$

$$\int \tan x dx = \ln \sec x; + C \qquad \int \cot x = \ln \sin x + C;$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C; \qquad \int \csc x dx = \ln|\csc x - \cot x| + C$$

$$\int \coth x dx$$

$$= \int \frac{\cosh x}{\sinh x} dx$$

$$= \int \frac{d \sinh x}{\sinh x} = \ln|\sinh x| + C$$

4. *Hyperbolic functions:*

$$\int \cosh x dx = \sinh x + C; \qquad \int \sinh x dx = \cosh x + C;$$

$$\int \tanh x dx = \ln \cosh x; + C \qquad \int \coth x = \ln \sinh x + C;$$

$$\int \text{sech} x dx = \tan^{-1} \sinh x + C; \qquad \int \text{csch} x dx = \ln|\text{csch} x - \coth x| + C$$

Proposition 2.2.4 (Arc length of graphs of functions).

1. *(Rectangular coordinates): The arc length of the curve given by the graph of function $y = f(x)$, $a < x < b$, in rectangular coordinates is*

$$l = \int_a^b \sqrt{1 + f'^2}\,dx.$$

2. *(Polar coordinates): The arc length of the curve given by the graph of function $r = r(\theta)$, $\alpha < \theta < \beta$, in polar coordinates is*

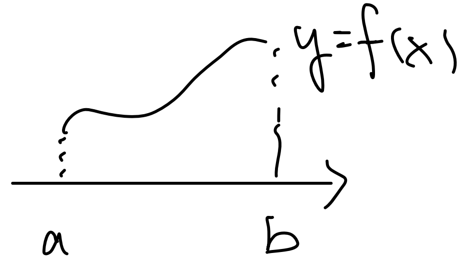
$$l = \int_\alpha^\beta \sqrt{r^2 + r'^2}\,d\theta.$$

displacement

$r(t) \quad a < t < b$

$$l = \int \underbrace{\|r'(t)\|}_{\text{speed}}\,dt$$

1

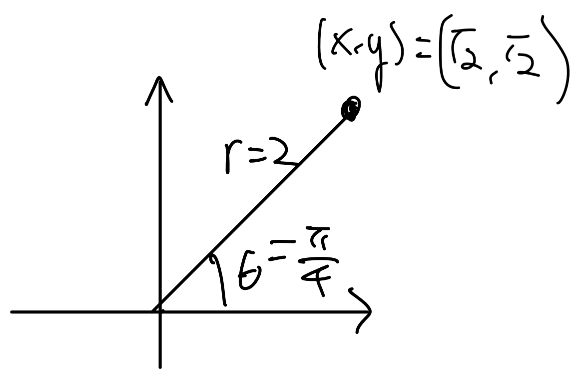


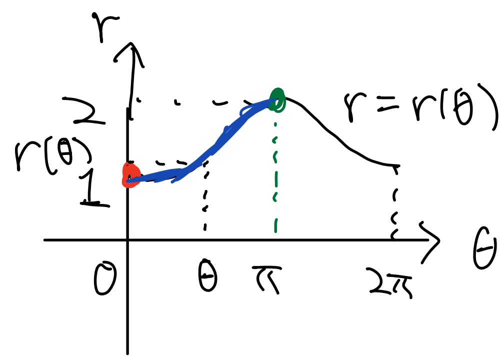
$$\vec{r}(x) = (x, f(x))$$

$$r'(x) = (1, f'(x))$$

$$\|r'(x)\| = \sqrt{1^2 + f'(x)^2}$$

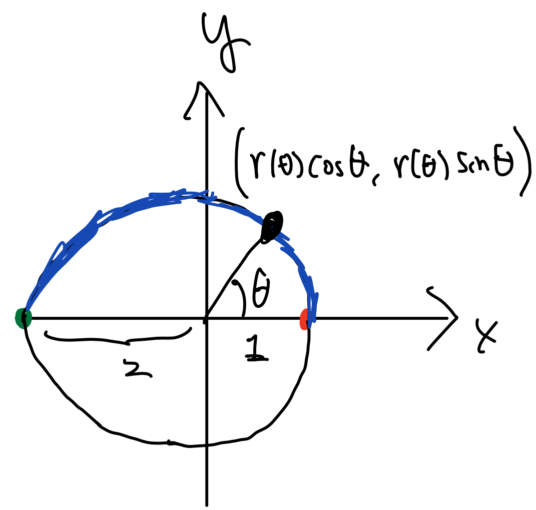
$$l = \int_a^b \|r'(x)\|\,dx = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$





Parametrization

$$\vec{r}(\theta) = (r(\theta)\cos\theta,\ r(\theta)\sin\theta)$$



$$l = \int_\alpha^\beta \sqrt{r^2 + r'^2}\, d\theta.$$

Pf $\quad \vec{r}'(\theta) = (r'\cos\theta - r\sin\theta,\ r'\sin\theta + r\cos\theta)$

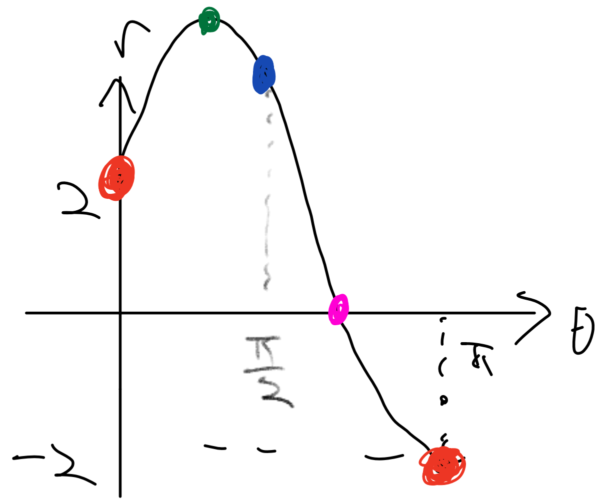
$$\|r'(\theta)\| = \sqrt{r'^2 + r^2}$$

eg $\quad r = 2\cos\theta + 4\sin\theta \qquad 0 < \theta < \pi$

$$r' = -2\sin\theta + 4\cos\theta$$

$$\sqrt{r^2 + (r')^2} = \sqrt{(2\cos\theta + 4\sin\theta)^2 + (-2\sin\theta + 4\cos\theta)^2}$$

$$= \sqrt{20}$$



$$l = \int_0^\pi \sqrt{20}\, d\theta = \sqrt{20}\,\pi$$

Rmk $\quad r = 2\cos\theta + 4\sin\theta$

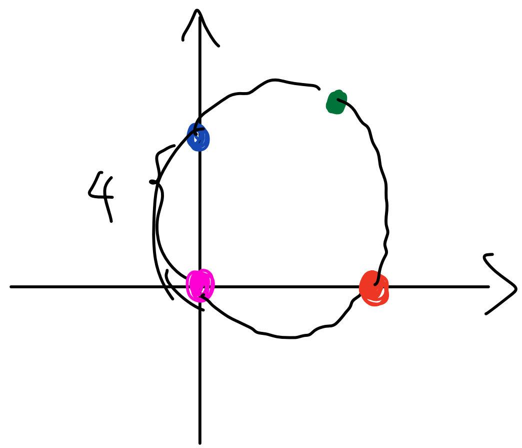
$$r^2 = 2r\cos\theta + 4r\sin\theta$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = 2x + 4y$$

$$(x-1)^2 + (y-2)^2 = \sqrt{5}^2 \quad \text{Circle}$$



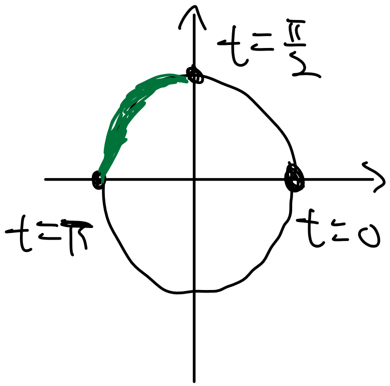
Rmk $\quad r = \sqrt{20}\sin(\theta + \alpha)$

Definition 2.2.5 (Arc length parametrization). *We say that* $\mathbf{r}(s)$ *is an* **arc length parametrized** *curve, or* $\mathbf{r}(s)$ *is* **parametrized by arc length**, *if* $\|\mathbf{r}'(s)\| = 1$ *for any* s.

$r(t) = (\cos t, \sin t)$

$r'(t) = (-\sin t, \cos t)$

$\|r'(t)\| = 1$ arc length parametrized

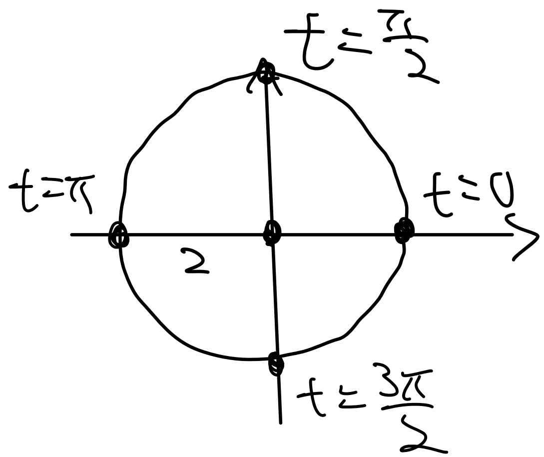


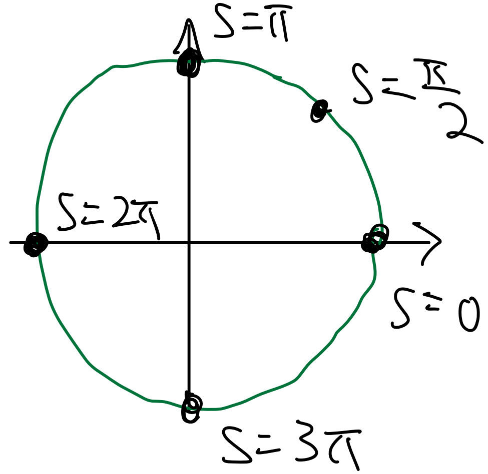
$r(t) = (2\cos t, 2\sin t)$ $\quad 0 \le t \le 2\pi$

$r'(t) = (-2\sin t, 2\cos t)$ $\quad \|r'(t)\| = 2$

$r(s) = \left(2\cos\frac{s}{2}, 2\sin\frac{s}{2}\right)$ $\quad \|r'(s)\| = 1$

$0 \le s \le 4\pi$

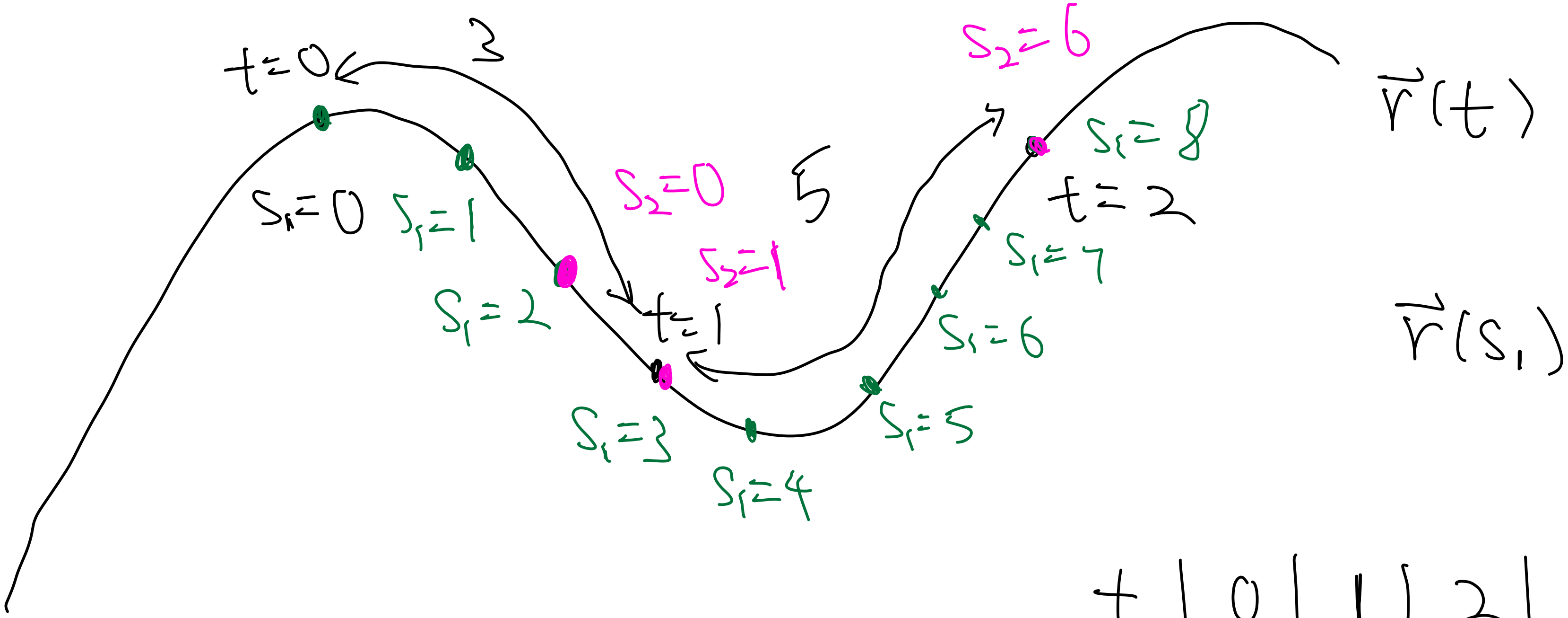




Proposition 2.2.6. *Let* $\mathbf{r}(s)$, *be an arc length parametrized curve. Then for* $a < b$, *the arc length of* $\mathbf{r}(s)$ *from* $s = a$ *to* $s = b$ *is* $b - a$.

Pf: $\quad \ell = \int_a^b \|r'(t)\|\,dt = \int_a^b 1\,dt = [t]_a^b = b - a$

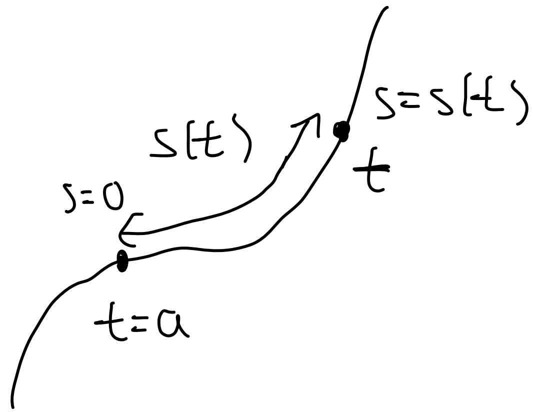
Theorem 2.2.7 (Existence and uniqueness of arc length parametrization). *Let* $\mathbf{r}(t)$ *be a regular parametrized curve. Then there exists increasing differentiable function* $s = s(t)$ *such that when* $\mathbf{r}(s)$ *is considered as a function of* s*, it is an arc length parametrized curve. Moreover if* $s_1(t)$ *and* $s_2(t)$ *are two such functions, then* $s_2 - s_1$ *is a constant.*



t	0	1	2
s_1	0	3	8
s_2	-2	1	6

<u>pf</u> See note

To find the arc length parametrization of $\mathbf{r}(t)$, we do the following three steps.



1. Find the arc length $s(t)$ as a function of t by

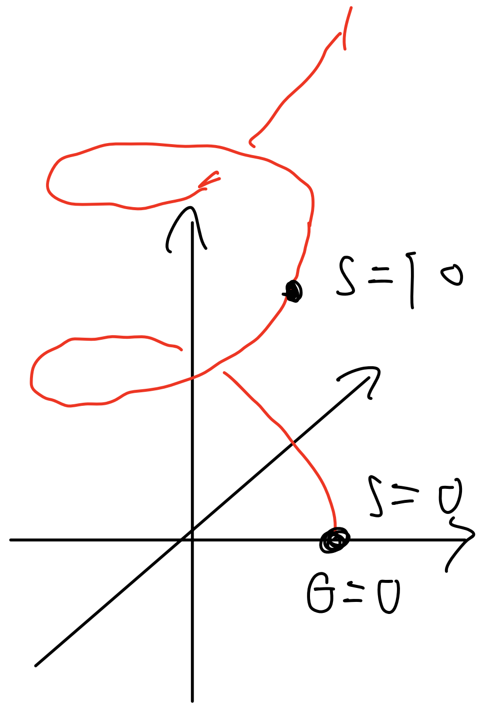
$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

2. Express $t = t(s)$ in terms of s. In other words, make t the subject in $s = s(t)$.

3. Substitute $t(s)$ into t in $\mathbf{r}(t)$ to get the arc length parametrization $\mathbf{r}(s)$.

Example 2.2.8 (Arc length parametrization of helix). *Let $a, b > 0$ be constants. Find an arc length parametrization of the helix $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$.*

1. $$s(\theta) = \int_0^\theta \|r'(u)\| du$$
$$= \int_0^\theta \sqrt{(-a\sin u)^2 + (a\cos u)^2 + b^2}\, du$$
$$= \int_0^\theta \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\,\theta$$ (s in terms of θ)



2. $$\theta = \frac{s}{\sqrt{a^2+b^2}}$$

3. $$r(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}} \right)$$

Example 2.2.9 (Arc length parametrization of catenary). *Find an arc length parametrization of the catenary* $\mathbf{r}(t) = (t, \cosh t)$.

$$\cosh t = \frac{e^t + e^{-t}}{2}$$

(1) $r'(t) = (1, \sinh t)$

$$s(t) = \int_0^t \|r'(u)\| du$$

$$= \int_0^t \sqrt{1^2 + (\sinh u)^2} du$$

$$= \int_0^t \cosh u \, du$$

$$= [\sinh x]_0^t = \sinh t$$

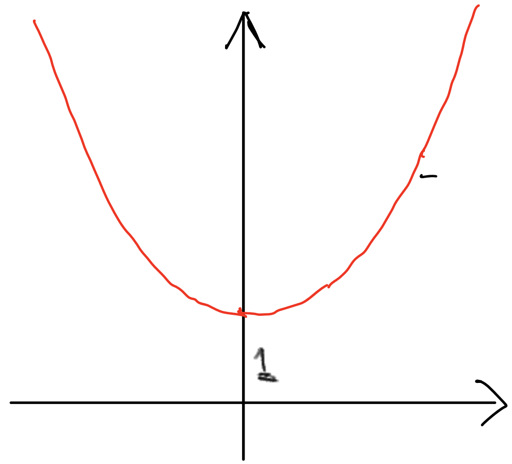
$$s = \frac{e^t - e^{-t}}{2}$$

$$2s = e^t - e^{-t}$$

$$2se^t = e^{2t} - 1$$

$$(e^t)^2 - 2se^t - 1 = 0$$

$$(e^t - s)^2 - s^2 - 1 = 0$$



(2) $t = \sinh^{-1} s = \ln(s + \sqrt{s^2+1}) \leftarrow e^t - s = \sqrt{s^2+1}$

(3) $r(s) = (\ln(s+\sqrt{s^2+1}), \cosh t)$

$$= (\ln(s+\sqrt{s^2+1}), \sqrt{1+(\sinh t)^2})$$

$$= (\ln(s+\sqrt{s^2+1}), \sqrt{1+s^2})$$

$$1 + (\sinh x)^2 = (\cosh x)^2$$

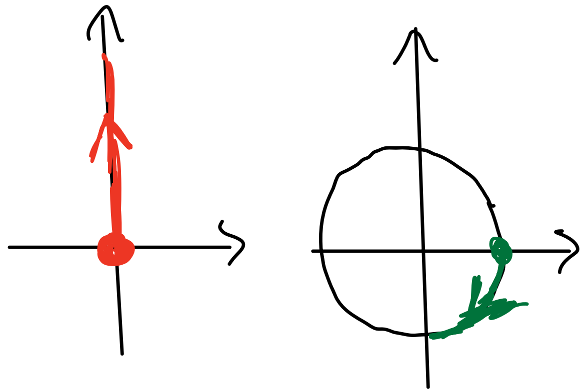
$$\int \cosh x \, dx = \sinh x + C$$

Example 2.2.10 (Tractrix). *The* **tractrix** *is a curve parametrized by*

$$\mathbf{r}(t) = (\operatorname{sech} t, t - \tanh t), \;\; t > 0.$$

$$= (0, t) + (\operatorname{sech} t, -\tanh t)$$

$$(\operatorname{sech} t)^2 + (\tanh t)^2 = 1$$



① $$r'(t) = (-\operatorname{sech} t \tanh t, 1 - \operatorname{sech}^2 t)$$
$$= (-\operatorname{sech} t \tanh t, \tanh^2 t)$$

$$\|r'(t)\|^2 = \operatorname{sech}^2 t \tanh^2 t + \tanh^4 t$$
$$= \tanh^2 t(\tanh^2 t + \operatorname{sech}^2 t)$$
$$= \tanh^2 t$$

$$\int \frac{\sinh u}{\cosh u}\, du = \int \frac{d \cosh u}{\cosh u}$$

$$S = \int_0^t \|r'(u)\|\, du$$
$$= \int_0^t \tanh u\, du$$
$$= \left[\ln \cosh u\right]_0^t = \ln \cosh t$$

② $$\cosh t = e^s$$

$$\vdots$$

$$r(s) = \left(e^{-s}, \ln\left(e^s + \sqrt{e^{2s} - 1}\right) - \sqrt{1 - e^{-2s}}\right)$$

Show

$$\frac{dy}{dx} = y' = -\frac{\sqrt{1 - x^2}}{x}$$

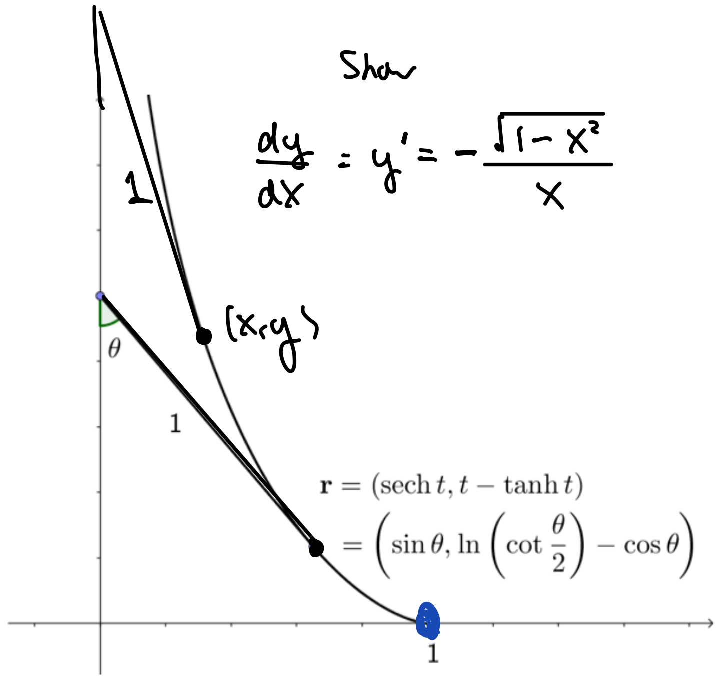


Figure 6: Tractrix

Note: The tractrix may also be parametrized by

$$\mathbf{r}(\theta) = \left(\sin\theta, \ln\left(\cot\frac{\theta}{2}\right) - \cos\theta\right), \ 0 < \theta < \frac{\pi}{2}.$$

Suppose L is the tangent to the tractrix at $\mathbf{r}(\theta)$ and P is the point of intersection of L and the y-axis. Then the angle between L and the y-axis is θ and the distance between $\mathbf{r}(\theta)$ and P is always 1.

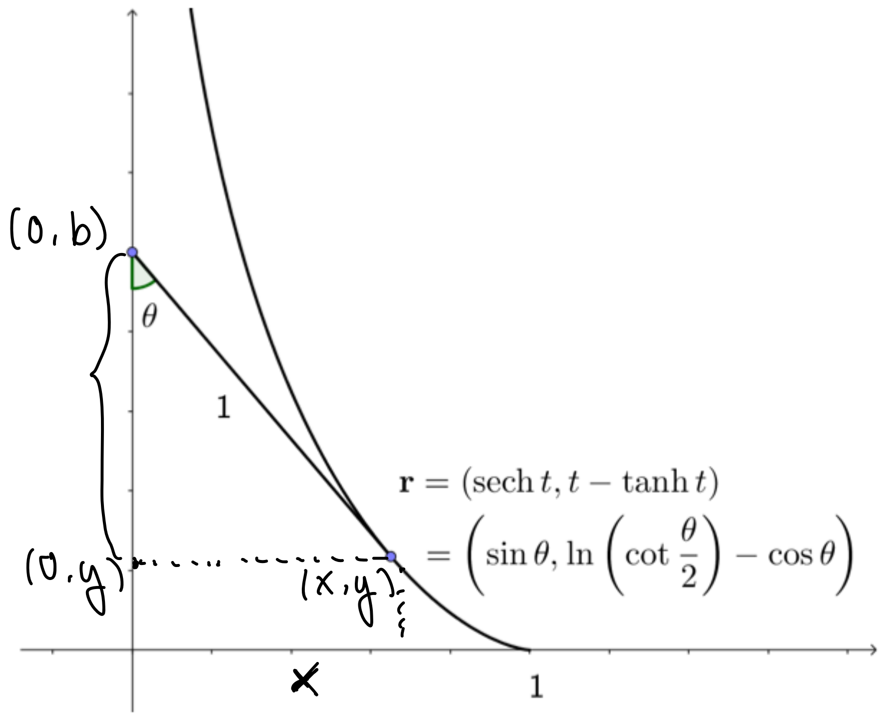


Figure 6: Tractrix

Three different parametrizations:

$$\mathbf{r}(t) = (\operatorname{sech} t, t - \tanh t), \ t > 0.$$

$$\mathbf{r}(s) = (e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \sqrt{1 - e^{-2s}}), \ s > 0.$$

$$\mathbf{r}(\theta) = \left(\sin\theta, \ln\left(\cot\frac{\theta}{2}\right) - \cos\theta\right), \ 0 < \theta < \frac{\pi}{2}.$$

To find $y = y(x)$ satisfying this:

$$x^2 + (y-b)^2 = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-b}{x} = -\frac{\sqrt{1-x^2}}{x}$$

Let $x = \sin\theta$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \cdot \cos\theta = -\frac{\cos^2\theta}{\sin\theta}$$

$$\frac{dy}{d\theta} = \frac{\sin^2\theta - 1}{\sin\theta} = \sin\theta - \csc\theta$$

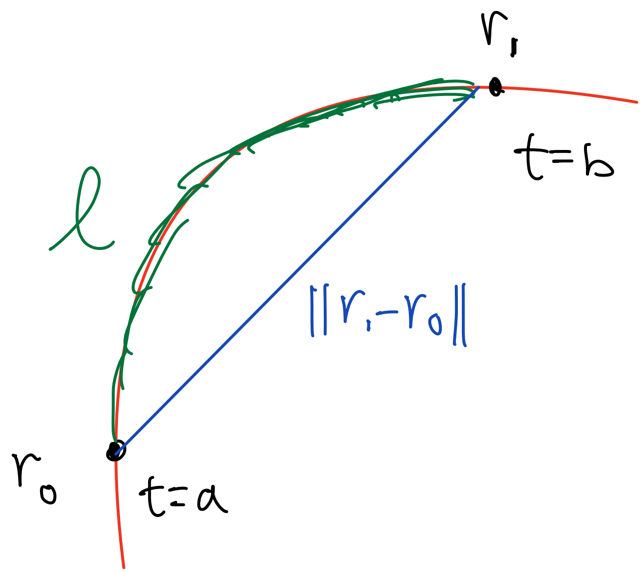
$$y = -\cos\theta - \ln\left|\tan\frac{\theta}{2}\right| + C$$

$y \to 0$ as $\theta \to \frac{\pi}{2} \Rightarrow C = 0$

Theorem 2.2.11. *Let $\mathbf{r}(t)$ be a regular parametrized curve with $\mathbf{r}(a) = \mathbf{r}_0$ and $\mathbf{r}(b) = \mathbf{r}_1$. Then the arc length l of the curve from $t = a$ to $t = b$ satisfies*

$$l \geq \|\mathbf{r}_1 - \mathbf{r}_0\|$$

with equality holds if and only if $\mathbf{r}(t)$ is a line segment joining $\mathbf{r}_0$ and $\mathbf{r}_1$.



Pf of inequality

let $\vec{u} = \dfrac{r_1 - r_0}{\|r_1 - r_0\|}$ unit vector fm r_0 to r_1

$$\begin{aligned}
l &= \int_a^b \|r'(t)\|\,dt \\
&\geq \int_a^b \langle r'(t), u\rangle\,dt \\
&= \int_a^b \langle r(t), u\rangle'\,dt \\
&= \Big[\langle r(t), u\rangle\Big]_a^b \\
&= \langle r(b) - r(a), u\rangle \\
&= \left\langle r_1 - r_0, \frac{r_1 - r_0}{\|r_1 - r_0\|}\right\rangle = \|r_1 - r_0\|
\end{aligned}$$

Cauchy-Schwarz Inequality

$$|\langle \vec{a}, \vec{b}\rangle| \leq \|\vec{a}\|\,\|\vec{b}\|$$

$$\langle r'(t), u\rangle \leq \|r'(t)\|\,\|u\| = \|r'(t)\|$$